Lec3; 01/26/2010

Homogeneous and isotropic spaces in 3 dimensions (Cont'd):

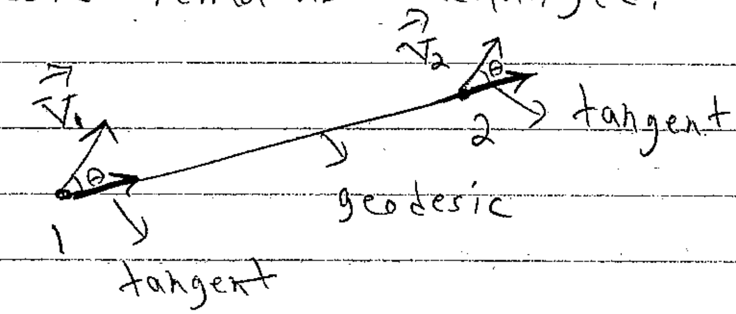
The three cases that we discussed have different geometries;

$ds^2 = a^2\left[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$ flat (no curvature)

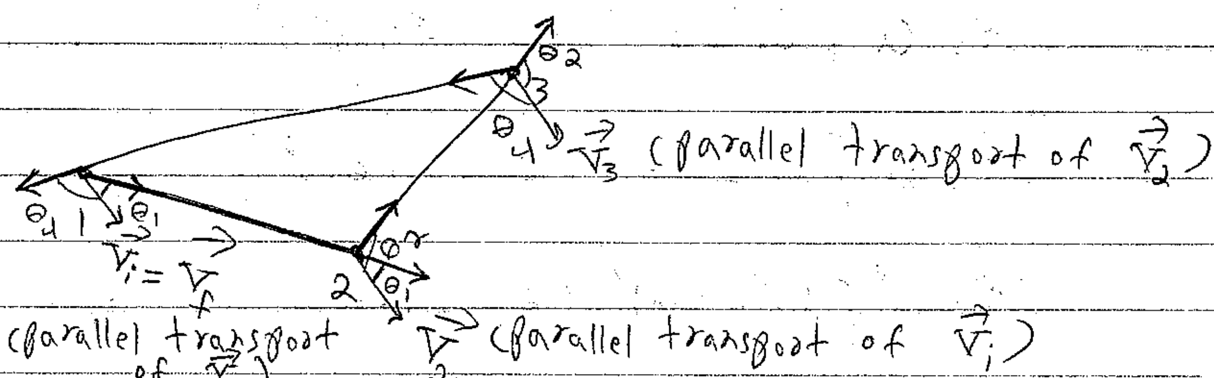
$ds^2 = a^2\left[dr^2 + \sin^2 r(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$ closed (+vely curved)

$ds^2 = a^2\left[dr^2 + \sinh^2 r(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$ open (-vely curved)

Here curved vs uncurved refers to the intrinsic curvature. A space has intrinsic curvature if parallel transport of a vector along a closed path consisting of geodesics yields a different vector. To elucidate, consider R^3. In R^3 geodesics (the paths with shortest distance) are straight lines. Parallel transport of a vector along a geodesic happens when the angle between the vector and tangent to the geodesic remains unchanged:



Now consider a closed path as follows:

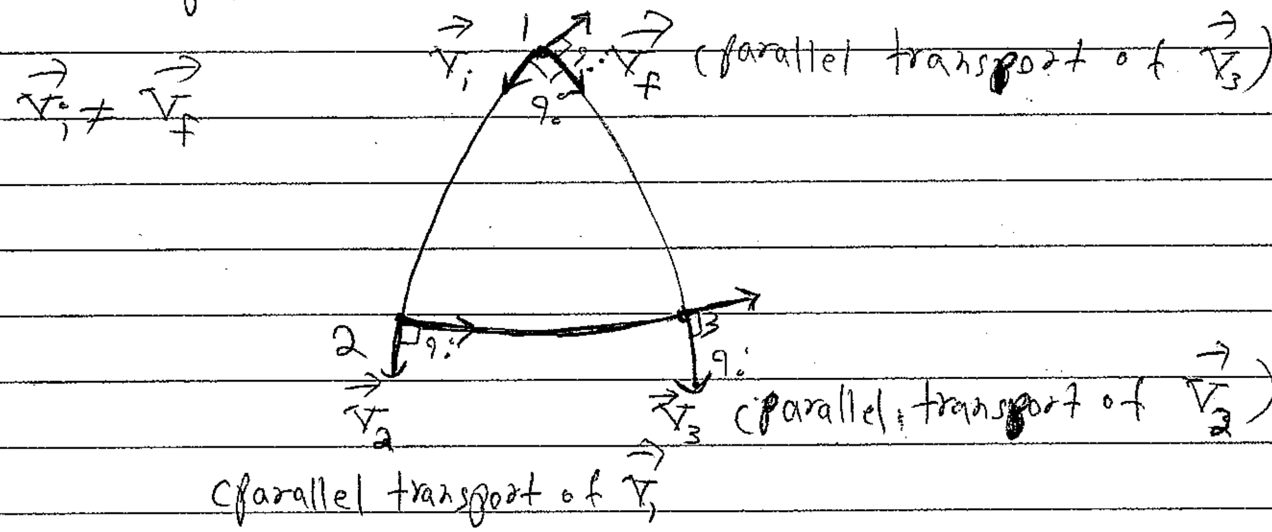


It is clear that $\vec{V}_i = \vec{V}_f$. This happens to be the case for any closed path that is likewise. Therefore, R^3 is flat and has no intrinsic curvature.

Next we consider a closed space. For simplicity, we consider S^2 instead of S^3. Lets the initial point be the north pole. Points 2 and 3 are on the equator and separated by an angle 90°. The vector $\vec{V}_i$ is tangent to the great circle connecting

1 and 2.

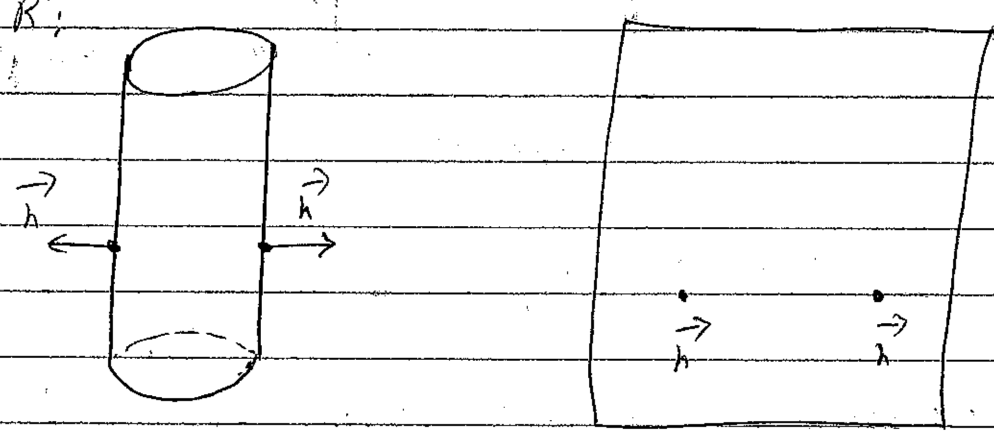
We note that geodesics in S^2 are great vectors on the sphere. We therefore have the following closed path:



We see that $\vec{v}_i \neq \vec{v}_f$, and hence S^2 has an intrinsic curvature (similarly for S^3). This is the same for H^3 (open universe).

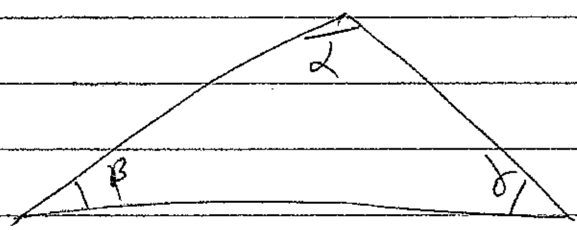
An important point to note is that the intrinsic curvature of a space can be figured out by living in that space only (using the above prescription).

However, there are spaces that have no intrinsic curvature but are curved when embedded in a higher dimensional space. For example, consider a cylinder. Its surface is the same as that of a plane in that it has no intrinsic curvature. But the difference arises after embedding in R^3:



As seen, the normal to the cylinder's surface does not have the same direction at all points. On the other hand, the normal to the plane has the same direction everywhere on the plane. The cylinder thus has extrinsic curvature, while the extrinsic curvature of the plane is zero (both have no intrinsic curvature)

Another way to see the intrinsic curvature of a 3-dimensional space is to consider triangles in these spaces. Note that each side of a triangle is a geodesic:



The space is flat (has no intrinsic curvature) if $\alpha+\beta+\gamma=\pi$. If $\alpha+\beta+\gamma>\pi$, then the space is positively curved. On the other hand, a space for which $\alpha+\beta+\gamma<\pi$ has negative curvature.

A look at the figure on page 20 shows that S^2 is positively curved (and so is S^3). While, H^3 is negatively curved.